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'Instantaneous source' solutions to a singular nonlinear diffusion equation

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Abstract. The nonlinear diffusion equation $u_t = (u^m u_x)_x$ possesses an instantaneous source similarity solution only for $m > -2$. Here we discuss physically motivated initial-boundary value problems for which a solution exists for all values of m. For delta function initial conditions, the case $m < -2$ is characterised by persistence of the delta function for a finite time.

1. Introduction

It is well known that for $m > -2$ the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right) \tag{1.1}$$

has an instantaneous source similarity solution corresponding to conditions

$$\left.\begin{aligned} &\text{as } |x| \to \infty \quad u \to 0,\ u^m \frac{\partial u}{\partial x} \to 0\,, \\ &\text{at } t = 0 \qquad u = M\delta(x)\,. \end{aligned}\right\} \tag{1.2}$$

For $m = 0$ we have

$$u = Mt^{-1/2} f(x/t^{1/2})\,, \tag{1.3}$$

where

$$f(\eta) = \frac{1}{2\sqrt{\pi}}\, \mathrm{e}^{-\eta^2/4}\,, \tag{1.4}$$

while for $m \neq 0$ we write

$$u = a^{2/m} t^{-1/(m+2)} f(x/at^{1/(m+2)})\,, \tag{1.5}$$

where the constant a is determined from

$$a^{(m+2)/m} \int_{-\infty}^{\infty} f(\eta)\, \mathrm{d}\eta = M\,.$$

For $m > 0$ we have (see Barenblatt [1])

$$\left.\begin{aligned} f(\eta) &= \left(\frac{m}{2(m+2)}(1-\eta^2)\right)^{1/m} & |\eta| < 1\,, \\ &= 0 & |\eta| \geq 1\,, \end{aligned}\right\} \tag{1.6}$$

so that the solution has compact support. When discussing the case $m < 0$ we shall write $n = -m$. For $0 < n < 2$ we have (see Landau and Lifschitz [2], p. 203)

$$f(\eta) = \left(\frac{n}{2(2-n)}(1+\eta^2)\right)^{-1/n}. \tag{1.7}$$

For $m \leq -2$, however, it is known that (1.1) subject to (1.2) has no solution; see, for example, Herrero [3] and Esteban et al. [4]. In this paper we shall focus on the case $m \leq -2$, though new results for $m > -2$ are also included for completeness, and we shall discuss some modifications to (1.2) which are suggested by physical considerations and for which (1.1) has a solution for all values of m. Our motivation comes from models for impurity diffusion in semiconductors. For example, models for gold diffusion in silicon lead under appropriate limits to (1.1) with $m = -2$ (Gösele et al. [5]) while zinc diffusion in gallium arsenide can lead to (1.1) with $m = -4$ (Gösele and Morehead [6]).

We consider two sets of boundary and initial conditions, namely the finite domain problem

$$\left.\begin{aligned} &\text{at } x = -L_1 & \frac{\partial u}{\partial x} &= 0\,, \\ &\text{at } x = L_2 & \frac{\partial u}{\partial x} &= 0\,, \\ &\text{at } t = 0 & u &= M\delta(x)\,, \end{aligned}\right\} \tag{1.8}$$

where L_1 and L_2 are positive constants, and the infinite domain problem

$$\left.\begin{aligned} &\text{as } |x| \to \infty & u &\to \varepsilon\,, \\ &\text{at } t = 0 & u &= M\delta(x) + \varepsilon\,, \end{aligned}\right\} \tag{1.9}$$

for some constant $\varepsilon > 0$. The conditions (1.9) either account for a non-zero background concentration, or correspond to a model problem

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left((c+\varepsilon)^m \frac{\partial c}{\partial x}\right)$$

(where $c = u - \varepsilon$) in which the diffusivity

$$D(c) = (c+\varepsilon)^m$$

does not blow up or vanish as $c \to 0$.

We note that for (1.8) it is possible to rescale to take, without loss of generality, $M = 1$ and either L_1 or $L_2 = 1$. Similarly, in (1.9) we may rescale to set $M = 1$. Here we prefer to retain these parameters explicitly since they represent quantities which can in practice be varied independently, and it is helpful to be able to observe the dependence on each of them directly. We note, however, that the parameter dependencies take the following forms.

(*a*) (1.1) *subject to* (1.8)

$$u = \frac{L_2}{M}\bar{u}\left(\frac{x}{L_2}, \frac{M^m t}{L_2^{m+2}}; m, \frac{L_1}{L_2}\right);$$

(*b*) (1.1) *subject to* (1.9)

$$u = \varepsilon\bar{u}\left(\frac{\varepsilon x}{M}, \frac{\varepsilon^{m+2} t}{M^2}; m\right).$$

We shall show that (1.1) subject to either (1.8) or (1.9) possesses a solution for all values of m. While the initial conditions are artificial, they will enable us to construct solutions which are in certain cases exact for some finite range of t, and our solutions provide indications of the behaviour in more general cases. Rigorous results for nonlinear diffusion equations which involve measures in the initial conditions have been given in, for example, Brezis and Friedman [7].

Our main tools are similarity methods and a non-local transformation originally due to Storm [8] (see also King [9] and references therein), together with singular perturbation methods. We first discuss (1.8) and then (1.9), after which we consider the behaviour for more general initial conditions. We conclude with some discussion.

2. Finite domain problems

2.1. $m \geq 0$

The behaviour of (1.1) subject to (1.8) is well understood for $m \geq 0$. For $m > 0$, expressions (1.5) and (1.6) give the solution exactly for

$$t < [\min(L_1, L_2)/a]^{m+2};$$

for larger t the solution cannot be found analytically. For $m = 0$ the solution is

$$u = \frac{M}{2\sqrt{\pi t}} \sum_{j=-\infty}^{\infty} \left(e^{-(x-2j(L_1+L_2))^2/4t} + e^{-(x-2j(L_1+L_2)-2L_2)^2/4t}\right). \tag{2.1}$$

It is instructive to consider the behaviour of (2.1) for small t. As $t \to 0^+$, equation (2.1) implies that

$$u \sim \frac{M}{2\sqrt{\pi t}} e^{-x^2/4t}$$

(which is the instantaneous source solution (1.3) and (1.4)), except near $x = -L_1$ and $x = L_2$; for $L_2 - x = O(t)$ we have

$$u \sim \frac{M}{\sqrt{\pi t}} e^{-L_2^2/4t} \cosh(L_2(L_2 - x)/2t),$$

with similar behaviour close to $x = -L_1$.

2.2. $0<n<2$

In this case the solution cannot be obtained exactly for finite t because the instantaneous source solution (1.5) and (1.7) does not have compact support. However the instantaneous source solution does describe the small-time behaviour close to $x=0$ and we may write

$$u \sim a^{-2/n} t^{-1/(2-n)} f(x/at^{1/(2-n)}) \quad \text{as } t \to 0^+ \text{ for } x = O(t^{1/(2-n)}),$$

where $f(\eta)$ is given by (1.7). This solution implies that

$$u \sim \left(\frac{n}{2(2-n)} \frac{x^2}{t}\right)^{-1/n}$$

for $t^{1/(2-n)} \ll |x| \ll 1$, and the solution in $x = O(1)$ is therefore given by the separable solution

$$u \sim t^{1/n} g(x) \quad \text{as } t \to 0^+$$

with

$$\left.\begin{aligned} &\frac{1}{n} g = \frac{\mathrm{d}}{\mathrm{d}x}\left(g^{-n} \frac{\mathrm{d}g}{\mathrm{d}x}\right), \\ &\text{as } x \to 0^+ \quad g \sim \left(\frac{n}{2(2-n)} x^2\right)^{-1/n}, \\ &\text{at } x = L_2 \quad \frac{\mathrm{d}g}{\mathrm{d}x} = 0, \end{aligned}\right\} \tag{2.2}$$

for $x>0$, and similarly for $x<0$. The solution to (2.2) may be written in the form

$$\int_{g/g_0}^{\infty} \omega^{-n}(\omega^{2-n}-1)^{-1/2}\,\mathrm{d}\omega = (2g_0^n/n(2-n))^{1/2}\, x \tag{2.3}$$

where

$$g_0 \equiv g(L_2) = (n(2-n)/2)^{1/n} \left(\int_1^{\infty} \omega^{-n}(\omega^{2-n}-1)^{-1/2}\,\mathrm{d}\omega / L_2\right)^{\frac{2}{n}}.$$

This solution can be illustrated by the special cases

$$n=1 \qquad g = \pi^2/2L_2^2 \sin^2\left(\frac{\pi x}{2L_2}\right),$$

$$n=\frac{4}{3} \qquad g = x^{-3/2}\left(1-\frac{x}{2L_2}\right)^{-3/2}.$$

2.3. $n \geqslant 2$

2.3.1. Reformulation

Turning now to the case $m \leqslant -2$ we shall use a non-local transformations to map the problem into one which is well understood (cf. Berryman [10]). We introduce

$$v = \int_{-L_1}^{x} u(x', t)\,\mathrm{d}x' ,$$

to give

$$\left.\begin{array}{ll} \dfrac{\partial v}{\partial t} = \left(\dfrac{\partial v}{\partial x}\right)^{-n} \dfrac{\partial^2 v}{\partial x^2} , & \\ \text{at } x = -L_1 & v = 0 , \\ \text{at } x = L_2 & v = M , \\ \text{at } t = 0, x < 0 & v = MH(x) , \end{array}\right\} \tag{2.4}$$

where $H(x)$ denotes the Heaviside step function, and then make use of the hodograph transformation

$$V = x, X = v, T = t$$

to give

$$\left.\begin{array}{ll} \dfrac{\partial V}{\partial T} = \left(\dfrac{\partial V}{\partial X}\right)^{n-2} \dfrac{\partial^2 V}{\partial X^2} , & \\ \text{at } X = 0 & V = -L_1 , \\ \text{at } X = M & V = L_2 , \\ \text{at } T = 0, 0 < X < M & V = 0 . \end{array}\right\} \tag{2.5}$$

Writing $U = \dfrac{\partial V}{\partial X}$ now gives

$$\left.\begin{array}{ll} \dfrac{\partial U}{\partial T} = \dfrac{\partial}{\partial X}\left(U^{n-2} \dfrac{\partial U}{\partial X}\right) , & \\ \text{at } X = 0 & \dfrac{\partial U}{\partial X} = 0 , \\ \text{at } X = M & \dfrac{\partial U}{\partial X} = 0 , \\ \text{at } T = 0, 0 < X < M & U = 0 , \\ \text{with } \displaystyle\int_0^X U(X', 0)\,\mathrm{d}X' = L_1, \int_X^M U(X', 0)\,\mathrm{d}X' = L_2 \text{ for } 0 < X < M , & \end{array}\right\} \tag{2.6}$$

so that we can map the problem (1.1) and (1.8) into a similar one in which the initial data now has a delta function at each of the ends $X = 0$ and $X = M$. Having determined the solution to (2.6), the solution to (1.1) and (1.8) is given by

$$u = 1/U, x = -L_1 + \int_0^X U(X', T)\,\mathrm{d}X', t = T . \tag{2.7}$$

As an aside, we note that in the limit $L_2 \to 0$ (or $L_1 \to 0$) the two problems (2.6) and (1.1) subject to (1.8) take essentially the same forms. Moreover, when $n = 1$ the exponents $-n$ and $n - 2$ are also the same so that if, in the limit $L_2 \to 0$, we write the solution to (2.4) with

$n=1$ in the form

$$v = M\Phi(x/L_1, t/ML_1) \tag{2.8}$$

then the solution to (2.5) is

$$V = -L_1\Phi(X/-M, T/ML_1),$$

which is

$$x = -L_1\Phi(v/-M, t/ML_1). \tag{2.9}$$

Equations (2.8) and (2.9) imply that if the solution is written in the form

$$\frac{v}{M} - \frac{x}{L_1} = \Psi\left(\frac{v}{M} + \frac{x}{L_1}, \frac{t}{ML_1}\right),$$

then Ψ satisfies the symmetry condition

$$\Psi(\xi, \tau) = \Psi(-\xi, \tau),$$

which implies in particular that $u = M/L_1$ (its average value) at the point at which $v/M = -x/L_1$.

Returning now to (2.6) we note that if $n=2$ the problem is linear, while if $n>2$ the exponent $n-2$ is positive.

2.3.2. $n=2$

If $n=2$ the solution to (2.6) is given by

$$U = \frac{1}{\sqrt{\pi T}}\left(L_1 \sum_{j=-\infty}^{\infty} e^{-(X-2jM)^2/4T} + L_2 \sum_{j=-\infty}^{\infty} e^{-(X-(2j+1)M)^2/4T}\right)$$

so that

$$V = -L_1 \sum_{j=0}^{\infty}\left(\operatorname{erfc}\left(\frac{X+2jM}{2T^{1/2}}\right) - \operatorname{erfc}\left(\frac{X-2jM}{2T^{1/2}}\right)\right)$$
$$+ L_2 \sum_{j=0}^{\infty}\left(\operatorname{erfc}\left(\frac{(2j+1)M-X}{2T^{1/2}}\right) - \operatorname{erfc}\left(\frac{(2j+1)M+X}{2T^{1/2}}\right)\right).$$

Using (2.7), the small time behaviour of the solution for u may be expressed as follows.

(1) $v = O(t^{1/2})$ $(x<0)$

Then

$$x \sim -L_1 \operatorname{erfc}(v/2t^{1/2}), \; u \sim (\pi t)^{1/2} e^{v^2/4t}/L_1,$$

so that

$$x \sim -L_1 \operatorname{erfc}(\ln^{1/2}(L_1 u/\sqrt{\pi t})). \tag{2.10}$$

(2) $M - v = O(t^{1/2})$ $(x > 0)$
Then

$$x \sim L_2 \operatorname{erfc}((M-v)/2t^{1/2}),\; u \sim (\pi t)^{1/2}\, e^{(M-v)^2/4t}/L_2\,,$$

so that

$$x \sim L_2 \operatorname{erfc}(\ln^{1/2}(L_2 u/\sqrt{\pi t}))\,. \tag{2.11}$$

Equations (2.10) and (2.11) both correspond to separable solutions so that

$$u \sim t^{1/2} g(x) \quad \text{as } t \to 0^+\,,$$

and $u = t^{1/2}g(x)$ exactly satisfies (1.1) with $m = -2$. We note that

$$\begin{aligned} &\text{at } x = -L_1 \quad u \sim (\pi t)^{1/2}/L_1\,,\\ &\text{at } x = L_2 \qquad u \sim (\pi t)^{1/2}/L_2\,. \end{aligned}$$

Equations (2.10) and (2.11) also imply that

$$\left.\begin{aligned} &\text{as } x \to 0^- \quad u \sim -t^{1/2}/x \ln^{1/2}(-1/x)\,,\\ &\text{as } x \to 0^+ \quad u \sim t^{1/2}/x \ln^{1/2}(1/x)\,. \end{aligned}\right\} \tag{2.12}$$

(3) $v = M/2 + O(t)$ $(x = O(t^{1/2}\exp(-M^2/16t)))$
In this case

$$\begin{aligned} x &\sim \frac{8(L_1L_2t)^{1/2}}{\pi^{1/2}Q}\, e^{-M^2/16t} \sinh\!\left(\frac{M(v-M/2)}{4t} + \tfrac{1}{2}\ln\!\left(\frac{L_2}{L_1}\right)\right),\\ u &\sim (\pi t)^{1/2}\, e^{M^2/16t}/2(L_1L_2)^{1/2} \cosh\!\left(\frac{M(v-M/2)}{4t} + \tfrac{1}{2}\ln\!\left(\frac{L_2}{L_1}\right)\right), \end{aligned}$$

giving

$$u \sim 4t/(M^2x^2 + 64L_1L_2t\, e^{-M^2/8t}/\pi)^{1/2}\,, \tag{2.13}$$

which holds as $t \to 0^+$ for $x = O(t^{1/2}\, e^{-M^2/16t})$. The expression (2.13) does not give an exact solution to (1.1) with $m = -2$, but does describe the asymptotic behaviour for small t.

We note that the separable solution plays the same role as for $0 < n < 2$, but the instantaneous source solution is replaced by (2.13), equation (1.1) having no instantaneous source similarity solution when $m = -2$.

We also note from (2.13) that

$$\text{at } x = 0 \quad u \sim (\pi t)^{1/2}\, e^{M^2/16t}/2(L_1L_2)^{1/2}\,,$$

so that for small t the maximum concentration is exponentially large.

2.3.3. $n>2$

The solution to (2.6) with $n>2$ may be written down exactly for finite T in the form

$$U = T^{-1/n}(a_1^{2/(n-2)} F(\eta_1) + a_2^{2/(n-2)} F(\eta_2)) , \tag{2.14}$$

where

$$\eta_1 = X/a_1 T^{1/n} , \quad \eta_2 = (M-X)/a_2 T^{1/n} ,$$

the constants a_1 and a_2 are given by

$$a_k^{n/(n-2)} \int_0^1 F(\eta)\,\mathrm{d}\eta = L_k , \quad k = 1, 2 , \tag{2.15}$$

and

$$\begin{aligned} F(\eta) &= \left(\frac{n-2}{2n}(1-\eta^2)\right)^{1/(n-2)} && 0 \leqslant \eta < 1 \\ &= 0 && \eta \geqslant 1 . \end{aligned}$$

The solution (2.14) is thus the sum of two instantaneous source solutions for U and is exact for

$$T < T_c ,$$

where

$$T_c \equiv (M/(a_1 + a_2))^n ; \tag{2.16}$$

we have

$$U = 0 \text{ for } a_1 T^{1/n} < X < M - a_2 T^{1/n} .$$

The form of solution for each of the variables arising in the non-local transformation is shown schematically in Fig. 1. Transforming this solution back we have

$$\text{at } x = 0 \quad u = (M - (a_1 + a_2) t^{1/n}) \delta(x) \quad \text{for } t < T_c ,$$

so that the delta function at the origin persists for finite time, though with diminishing magnitude. In each of $x<0$ and $x>0$ we have a separable solution

$$u = t^{1/n} g(x) \quad \text{for } t < T_c$$

into which the instantaneous source solutions for U map. In $x>0$ this may be written

$$\int_{g/g_0}^{\infty} \omega^{-n}(1-\omega^{2-n})^{-1/2}\,d\omega = (2g_0^n/n(n-2))^{1/2}x\,,$$

where

$$g_0 \equiv g(L_2) = (n(n-2)/2)^{1/n}\left(\int_1^{\infty} \omega^{-n}(1-\omega^{2-n})^{-1/2}\,d\omega/L_2\right)^{2/n}. \tag{2.17}$$

We note that $g \to +\infty$ as $x \to 0^+$, but in contrast to (2.2) the local behaviour is given by

$$g = O(x^{1/(1-n)}) \quad \text{as } x \to 0^+\,,$$

which represents a quasi-steady balance in (1.1).

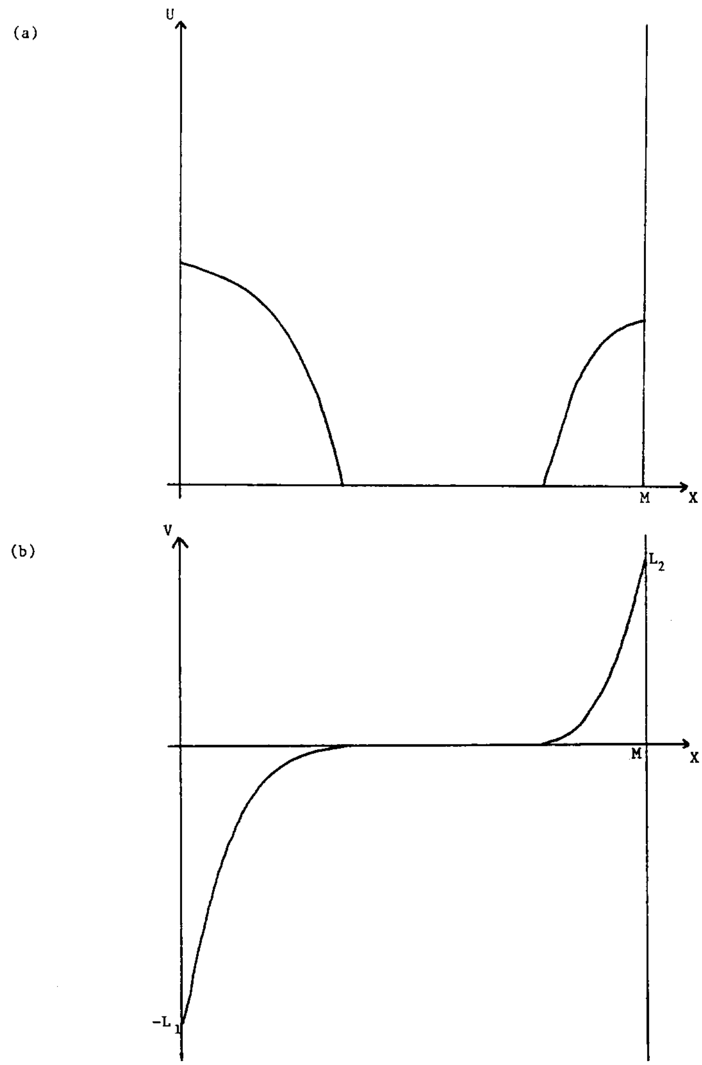
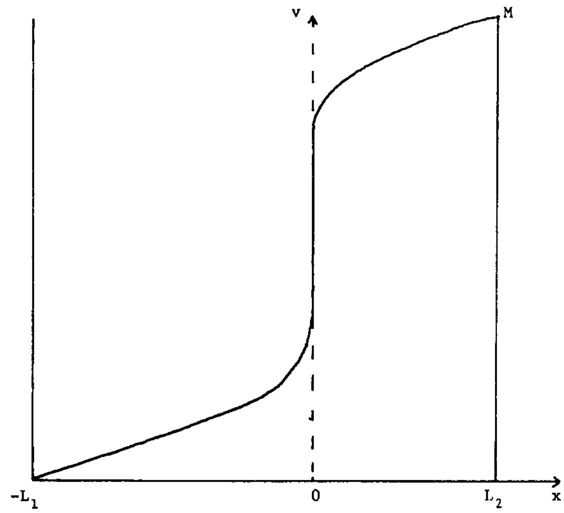


Fig. 1. Schematic of solutions for the finite domain problem for $0 < t < T_c$. (a) $U(X,T)$; see (2.6). (b) $V(X,T)$; see (2.5). (c) $v(x,t)$; see (2.4). (d) $u(x,t)$; see (1.1) and (1.8).

(c)



(d)

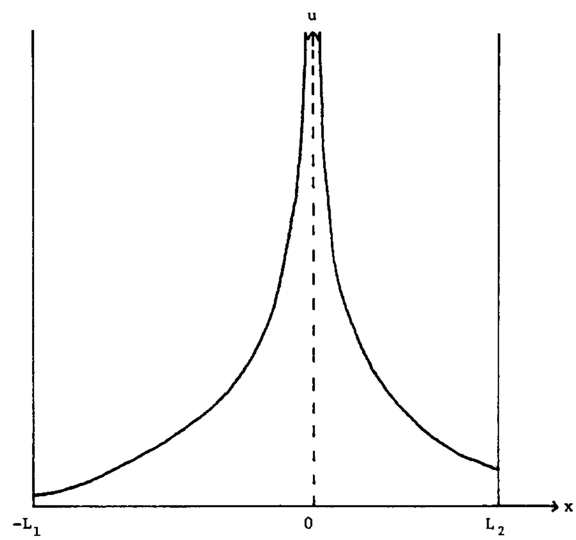


Fig. 1. (contd.).

The asymptotic behaviour close to $T = T_c$ may also be determined. Writing $X_c = a_1 T_c^{1/n}$ then we have at $T = T_c$

$$U \sim \left(\frac{n-2}{n} T_c^{-(n-1)/n} a_1 (X_c - X)\right)^{1/(n-2)} \quad \text{as } X \to X_c^- ,$$

and

$$U \sim \left(\frac{n-2}{n} T_c^{-(n-1)/n} a_2 (X - X_c)\right)^{1/(n-2)} \quad \text{as } X \to X_c^+ .$$

The behaviour near $X = X_c$ as $T \to T_c^+$ is then given by

$$U \sim (T - T_c)^{1/(n-2)} B(\chi) , \tag{2.18}$$

where

$$\chi = (X - X_c)/(T - T_c)$$

and $B(\chi)$ satisfies

$$\begin{aligned} &\frac{1}{n-2} B - \chi \frac{\mathrm{d}B}{\mathrm{d}\chi} = \frac{\mathrm{d}}{\mathrm{d}\chi}\left(B^{n-2} \frac{\mathrm{d}B}{\mathrm{d}\chi}\right), \\ &\text{as } \chi \to -\infty \quad B \sim \left(-\frac{n-2}{n} T_c^{-(n-1)/n} a_1 \chi\right)^{1/(n-2)}, \\ &\text{as } \chi \to +\infty \quad B \sim \left(\frac{n-2}{n} T_c^{-(n-1)/n} a_2 \chi\right)^{1/(n-2)}. \end{aligned} \tag{2.19}$$

If $B_{\min} \equiv \min(B)$ then, because $u = 1/U$, the maximum value of u satisfies

$$u_{\max} \sim (t - T_c)^{-1/(n-2)}/B_{\min} \quad \text{as } t \to T_c^+ . \tag{2.20}$$

We summarise our results for the finite domain problem by the following comments.

For $m > 0$ the solution (1.5) has compact support and does not feel the boundaries of the domain until some finite time. For $-2 < m < 0$ the small time behaviour is described by (1.5) close to $x = 0$, so the high concentration region in this case also initially ignores the presence of the boundaries; the low concentration behaviour is governed by the separable solution and is dependent on the location of the boundaries (see (2.2)).

For $n > 2$ the delta function persists for a finite time, but is depleted by the transfer of material into the low concentration regions, which can again be described by the separable solution. It follows from (2.15) and (2.16) that the time of extinction of the delta function, T_c, satisfies

$$T_c \propto M^n/(L_1^{(n-2)/n} + L_2^{(n-2)/n})^n .$$

Hence $T_c \to 0$ as L_1 or $L_2 \to \infty$ with M fixed; this occurs because the low concentration regions extract material from the delta function more rapidly as L_1 and L_2 increase.

3. Infinite domain problems

3.1. $m \geqslant 0$

We now discuss the behaviour of (1.1) subject to (1.9). Exploiting the symmetry of the problem we may consider

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right), \\ &\text{at } x = 0 \quad \frac{\partial u}{\partial x} = 0, \\ &\text{as } x \to +\infty \quad u \to \varepsilon, \\ &\text{at } t = 0 \quad u = M\delta(x) + \varepsilon ; \end{aligned}\right\} \tag{3.1}$$

we note that

$$\int_0^\infty (u-\varepsilon)\,\mathrm{d}x = M/2\,.$$

Although the problem may be rescaled to make $\varepsilon = 1$, for $m > -2$ we shall concentrate on the asymptotic behaviour for $\varepsilon \ll 1$, which is equivalent to considering the small time behaviour.

For $m > 0$ the leading order behaviour as $\varepsilon \to 0$ with $t = O(1)$ is given for $x < at^{1/(m+2)}$ by (1.5) and (1.6). The exact solution, however, does not have compact support, and in the limit $\varepsilon \to 0$ (3.1) is a singular perturbation problem. This problem has some interesting features from the point of view of asymptotics, but since it lies outside the main theme of this paper we relegate its discussion to Appendix 1.

For $m = 0$ the solution to (3.1) is

$$u = \frac{M}{2\sqrt{\pi t}}\,\mathrm{e}^{-x^2/4t} + \varepsilon\,.$$

3.2. $0 < n < 2$

The analysis for this case follows that of King [11]. Writing

$$u = u_0(x,t) + o(1) \quad \text{as } \varepsilon \to 0$$

for $x = O(1)$, the leading order solution is again

$$u_0 = a^{-2/n} t^{-1/(2-n)} f(x/at^{1/(2-n)})\,,$$

where $f(\eta)$ is given by (1.7). It follows that

$$u_0 \sim \left(\frac{n}{2(2-n)}\,\frac{x^2}{t}\right)^{-1/n} \qquad \text{as } x \to +\infty\,.$$

There is an outer region in which the rescalings are

$$x = \varepsilon^{-n/2} y \qquad u = \varepsilon w$$

and writing

$$w = w_0(y,t) + o(1) \quad \text{as } \varepsilon \to 0$$

we obtain

$$\left.\begin{aligned}
&\frac{\partial w_0}{\partial t} = \frac{\partial}{\partial y}\left(w_0^{-n}\frac{\partial w_0}{\partial y}\right), \\
&\text{as } y \to 0^+ \quad w_0 \sim \left(\frac{n}{2(2-n)}\,\frac{y^2}{t}\right)^{-1/n}, \\
&\text{as } y \to +\infty \quad w_0 \to 1\,, \\
&\text{at } t = 0 \qquad w_0 = 1\,.
\end{aligned}\right\} \tag{3.2}$$

The solution to (3.2) takes the self-similar form

$$w_0 = p(y/t^{\frac{1}{2}}),$$

where $p(\zeta)$ satisfies

$$\begin{aligned} &-\tfrac{1}{2}\zeta \frac{\mathrm{d}p}{\mathrm{d}\zeta} = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(p^{-n}\frac{\mathrm{d}p}{\mathrm{d}\zeta}\right), \\ &\text{as } \zeta \to 0^+ \quad p \sim \left(\frac{n}{2(2-n)}\zeta^2\right)^{-1/n}, \\ &\text{as } \zeta \to +\infty \quad p \to 1 . \end{aligned} \tag{3.3}$$

3.3. $n \geqslant 2$

3.3.1. Reformulation

For this case we are able to describe the behaviour exactly for some finite time. We do not therefore need to restrict attention to the small time behaviour, and we use the rescalings

$$u \to \varepsilon u, \quad x \to x/\varepsilon, \quad t \to t/\varepsilon^{m+2}$$

to obtain

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right), \\ &\text{at } x = 0 \qquad \frac{\partial u}{\partial x} = 0, \\ &\text{as } x \to +\infty \quad u \to 1, \\ &\text{at } t = 0 \qquad u = M\delta(x) + 1 . \end{aligned}\right\} \tag{3.4}$$

We introduce

$$v = \int_0^x u(x', t)\,\mathrm{d}x'$$

to give

$$\begin{aligned} &\frac{\partial v}{\partial t} = \left(\frac{\partial v}{\partial x}\right)^{-n} \frac{\partial^2 v}{\partial x^2}, \\ &\text{at } x = 0 \qquad v = 0, \\ &\text{as } x \to +\infty \quad v \sim x + M/2, \\ &\text{at } t = 0 \qquad v = x + M/2 . \end{aligned}$$

The hodograph transformation

$$V = x,\ X = v,\ T = t$$

now yields

$$\frac{\partial V}{\partial T}=\left(\frac{\partial V}{\partial X}\right)^{n-2}\frac{\partial^2 V}{\partial X^2},$$

$$\begin{aligned}&\text{as } X=0 && V=0\,,\\ &\text{as } X\to+\infty && V\sim X-M/2\,,\\ &\text{at } T=0 && V=\left(X-\frac{M}{2}\right)H\left(X-\frac{M}{2}\right).\end{aligned}$$

Finally, writing $U=\dfrac{\partial V}{\partial X}$ gives

$$\left.\begin{aligned}&\frac{\partial U}{\partial T}=\frac{\partial}{\partial X}\left(U^{n-2}\frac{\partial U}{\partial X}\right),\\ &\text{at } X=0 \qquad \frac{\partial U}{\partial X}=0\,,\\ &\text{as } X\to+\infty \quad U\to 1\,,\\ &\text{at } T=0 \qquad U=H\left(X-\frac{M}{2}\right).\end{aligned}\right\} \tag{3.5}$$

The solution to (3.4) is given in terms of the solution to (3.5) by

$$u=1/U\,,\quad x=\int_0^X U(X',T)\,\mathrm{d}X',\, t=T\,. \tag{3.6}$$

3.3.2. $n=2$
In this case (3.5) has solution

$$U=(\operatorname{erfc}((M-2X)/4T^{\frac{1}{2}})+\operatorname{erfc}((M+2X)/4T^{1/2}))/2\,, \tag{3.7}$$

so that

$$V=\Big((M+2X)\operatorname{erfc}((M+2X)/4T^{1/2})-(M-2X)\operatorname{erfc}((M-2X)/4T^{1/2})$$
$$+4\sqrt{\frac{T}{\pi}}\,(\mathrm{e}^{-(M-2X)^2/16T}-\mathrm{e}^{-(M+2X)^2/16T})\Big)\Big/4\,. \tag{3.8}$$

Using (3.6), the small t behaviour of the solution to (3.4) may then be determined as follows.

(1) $v=M/2+O(t^{1/2})$ $(x=O(t^{1/2}))$
Writing

$$v-M/2=-t^{1/2}\xi$$

gives the small t solution in the form

$$\left.\begin{aligned} x/t^{1/2} &\sim \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-\xi^2/4} - \frac{\xi}{2}\,\mathrm{erfc}(\xi/2)\,, \\ u &\sim 2/\mathrm{erfc}(\xi/2)\,. \end{aligned}\right\} \tag{3.9}$$

We note that this solution, which describes the small t behaviour everywhere except very close to $x = 0$, takes the form

$$u \sim p(x/t^{1/2})\,.$$

The solution given by (3.9) has

$$u \sim t^{1/2}/x \ln^{1/2}(1/x) \quad \text{as } x \to 0^+\,.$$

(2) $v = O(t)$ $(x = O[t^{3/2}\exp(-M^2/16t)])$
Now

$$x \sim 16t^{3/2}\,\mathrm{e}^{-M^2/16t}\sinh(Mv/4t)/\pi^{1/2}M^2\,,$$

$$u \sim \pi^{1/2}M\,\mathrm{e}^{M^2/16t}/4t^{1/2}\cosh(Mv/4t)\,,$$

so that

$$u \sim 4t/(M^2x^2 + 256t^3\,\mathrm{e}^{-M^2/8t}/\pi M^2)^{1/2}\,. \tag{3.10}$$

We have the exact result

$$\text{at } x = 0 \quad u = 1/\mathrm{erfc}(M/4t^{1/2})\,,$$

which implies that

$$\text{at } x = 0 \quad u \sim \pi^{1/2}M\,\mathrm{e}^{M^2/16t}/4t^{1/2} \quad \text{as } t \to 0^+\,,$$

which is again exponentially large.

3.3.3. $n > 2$
For $n > 2$ and for sufficiently small T the exact solution to (3.5) takes the form

$$U = P[(M - 2X)/2T^{1/2}]\,, \tag{3.11}$$

where $P(\xi)$ satisfies

$$\left.\begin{aligned} &-\tfrac{1}{2}\xi\,\frac{\mathrm{d}P}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(P^{n-2}\,\frac{\mathrm{d}P}{\mathrm{d}\xi}\right), \\ &\text{as } \xi \to -\infty \quad P \to 1\,, \\ &\text{at } \xi = \xi_0 \quad P = 0\,, \quad \lim_{\xi \to \xi_0}\left(P^{n-3}\,\frac{\mathrm{d}P}{\mathrm{d}\xi}\right) = -\tfrac{1}{2}\xi_0\,, \end{aligned}\right\} \tag{3.12}$$

where the constant ξ_0 is determined as part of the solution to (3.12). We then have

$$U = 0 \quad \text{for} \quad 0 \leqslant X \leqslant (M - 2\xi_0 T^{1/2})/2\,,$$

and the solution takes the form (3.11) for

$$T < T_c\,,$$

where, in this case,

$$T_c \equiv M^2/4\xi_0^2\,. \tag{3.13}$$

If we define

$$Q(\xi) = \int_\xi^{\xi_0} P(\xi')\,\mathrm{d}\xi'\,,$$

then the solution to (3.4) is given for $t < T_c$, $x > 0$ by

$$x = t^{1/2} Q(\xi)\,,$$

$$u = 1/P(\xi)\,,$$

so that u again takes the form

$$u = p(x/t^{1/2})$$

in $x > 0$. The function $p(\zeta)$ can alternatively be obtained directly as the solution to

$$\left.\begin{aligned} &-\tfrac{1}{2}\zeta\,\frac{\mathrm{d}p}{\mathrm{d}\zeta} = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(p^{-n}\,\frac{\mathrm{d}p}{\mathrm{d}\zeta}\right),\\ &\text{as } \zeta \to 0^+ \quad p \to +\infty\,,\\ &\text{as } \zeta \to +\infty \quad p \to 1\,. \end{aligned}\right\} \tag{3.14}$$

Since

$$P(\xi) \sim \left(\frac{(n-2)}{2}\,\xi_0(\xi_0 - \xi)\right)^{1/(n-2)} \quad \text{as } \xi \to \xi_0^-\,,$$

it follows that

$$p(\zeta) \sim \left(\frac{(n-1)}{2}\,\xi_0\zeta\right)^{-1/(n-1)} \quad \text{as } \zeta \to 0^+\,,$$

which may be contrasted with (3.3). We also note from (3.14) the integral results

$$\int_0^\infty (p(\zeta) - 1)\,\mathrm{d}\zeta = \xi_0\,,$$

and

$$\int_0^\infty \zeta(p(\zeta) - 1)\,\mathrm{d}\zeta = \frac{1}{n-1}\,;$$

the latter implies that

$$\int_0^\infty x(u(x,t) - 1)\,\mathrm{d}x = \frac{t}{n-1}\,.$$

At $x = 0$ we again have a delta function for sufficiently small t; in this case

$$\text{at } x = 0 \quad u = M(1 - (t/T_c)^{1/2})\delta(x) \quad \text{for } t < T_c\,. \tag{3.15}$$

The behaviour close to the time at which the delta function disappears is similar to that discussed in Section 2.3.3. Writing

$$\chi = X/(T - T_c)\,,$$

we then have

$$U \sim (T - T_c)^{1/(n-2)}\,B(\chi) \quad \text{as } T \to T_c^+ \text{ with } \chi = O(1)\,, \tag{3.16}$$

where $B(\chi)$ can be determined from

$$\frac{1}{n-2}\,B - \chi\,\frac{\mathrm{d}B}{\mathrm{d}\chi} = \frac{\mathrm{d}}{\mathrm{d}\chi}\left(B^{n-2}\,\frac{\mathrm{d}B}{\mathrm{d}\chi}\right),$$

$$\text{at } \chi = 0 \quad \frac{\mathrm{d}B}{\mathrm{d}\chi} = 0\,,$$

$$\text{as } \chi \to +\infty \quad B \sim [(n-2)\xi_0^2\chi/M]^{1/(n-2)}\,.$$

The following comments summarise our infinite domain results.

For $m > -2$ and for small time, the high concentration region ignores the presence of the small background concentration. The transition to this background concentration is of travelling wave type for $m > 0$ (see Appendix 1) and is governed by a Boltzmann similarity solution for $-2 < m < 0$ (see (3.3)).

For $n > 2$ the delta function again persists for finite time. In terms of the original time variable, the time of extinction of the delta function is proportional to $\varepsilon^{n-2}M^2$. For fixed M the delta function therefore disappears more rapidly as ε decreases; in the limit $\varepsilon \to 0$ the material instantaneously diffuses out to infinity.

4. General initial conditions

4.1. Formulation

In this section we shall consider the following generalisation of the infinite domain problem:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right), \\ &\text{as } |x| \to \infty \quad u \to \varepsilon, \\ &\text{at } t = 0 \qquad u = I(x) + \varepsilon, \end{aligned}\right\} \tag{4.1}$$

where $I(x)$ is some specified positive function which satisfies

$$\int_{-\infty}^{\infty} I \, \mathrm{d}x = M .$$

Results for the equivalent generalisation to the finite domain problem may be derived in a similar manner to those of this section.

We shall consider the case $\varepsilon \ll 1$, with $I(x) = O(1)$ for $x = O(1)$. The detailed structure depends on how $I(x)$ decays for large x, and for definiteness we shall assume that

$$I(x) \sim Ax^{-\alpha} \quad \text{as } x \to +\infty , \tag{4.2}$$

for constants $A > 0$ and $\alpha > 1$, with similar behaviour as $x \to -\infty$. We shall for the most part discuss the behaviour in $x > 0$ only; the behaviour in $x < 0$ follows in an obvious fashion.

When t becomes sufficiently large it is clear that u will drop to become $O(\varepsilon)$ everywhere, and (4.1) ceases to be amenable to asymptotic methods. Here we shall only be interested in the earlier stages of the development in which $u_{\max} \gg \varepsilon$. With regard to the late-stage behaviour we simply remark that for very large times we have

$$u - \varepsilon \sim \frac{M}{2(\pi\varepsilon^m t)^{1/2}} \, \mathrm{e}^{-x^2/4\varepsilon^m t} \quad \text{for } x = O(\varepsilon^{m/2} t^{1/2}) .$$

4.2. $m > -2$

Related results for $m = 1$ are given in [12] and [13], and the results of [11] are relevant when $-2 < m < 0$. Briefly, the asymptotic behaviour is as follows. In $x = O(1)$ we write for $t = O(1)$

$$u = u_0(x, t) + o(1) ,$$

with

$$\left.\begin{aligned} &\frac{\partial u_0}{\partial t} = \frac{\partial}{\partial x}\left(u_0^m \frac{\partial u_0}{\partial x}\right), \\ &\text{as } |x| \to \infty \quad u_0 \to 0, \\ &\text{at } t = 0 \qquad u_0 = I(x) . \end{aligned}\right\} \tag{4.3}$$

The large-time behaviour of (4.3) is given by (1.5) together with (1.6) or (1.7). We now discuss the cases $m > 0$ and $m < 0$ separately.

(a) $m > 0$

The far-field behaviour of (4.1) is steady-state for $m > 0$:

$$u \sim \varepsilon + Ax^{-\alpha} \quad \text{as } x \to +\infty .$$

The large t behaviour of (4.3) is described by (1.5) only for $x < at^{1/(m+2)}$. From (4.2) it follows that when x is close to $at^{1/(m+2)}$ then $I = O(t^{-\alpha/(m+2)})$ and this is of $O(\varepsilon)$ when $t = O(\varepsilon^{-(m+2)/\alpha})$ which is therefore an important timescale. Writing

$$t = \varepsilon^{-(m+2)/\alpha}\hat{t}, \quad x = \varepsilon^{-1/\alpha}\hat{y}, \quad u = \varepsilon^{1/\alpha}\hat{u}$$

gives

$$\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial}{\partial \hat{y}}\left(\hat{u}^m \frac{\partial \hat{u}}{\partial \hat{y}}\right).$$

We then have

$$\hat{u} \sim a^{2/m}\,\hat{t}^{-1/(m+2)} f(\hat{y}/a\hat{t}^{1/(m+2)}) \text{ for } \hat{y} < a\hat{t}^{1/(m+2)}, \tag{4.4}$$

where f is given by (1.6), and near $\hat{y} = a\hat{t}^{1/(m+2)}$ we write

$$\hat{y} = \hat{s}(\hat{t};\, \varepsilon) + \varepsilon^{m(1-1/\alpha)}\hat{z}, \quad \hat{u} = \varepsilon^{1-1/\alpha}w,$$

where $\hat{s}(\hat{t};\, 0) = a\hat{t}^{1/(m+2)}$, to give at leading order

$$-\frac{1}{m+2}\, a\hat{t}^{-(m+1)/(m+2)}(w_0 - 1 - Aa^{-\alpha}\hat{t}^{-\alpha/(m+2)}) = w_0^m \frac{\partial w_0}{\partial \hat{z}} \tag{4.5}$$

(compare (A1.1)). In $\hat{y} > a\hat{t}^{1/(m+2)}$

$$w \sim 1 + A\hat{y}^{-\alpha} \tag{4.6}$$

holds. Equation (4.5) describes the narrow region of transition between the instantaneous source solution (4.4) and the far-field behaviour (4.6).

On the even longer timescale $t = O(\varepsilon^{-(m+2)})$, u is of $O(\varepsilon)$ everywhere.

(*b*) $0 < n < 2$

The far-field behaviour of the solution to (4.3) may now be categorised as follows.

(i) $1 < \alpha < \frac{2}{n}$ $\quad u_0 \sim Ax^{-\alpha} \quad$ as $x \to +\infty$;

(ii) $\alpha = \frac{2}{n}$ $\quad u_0 \sim \left(\frac{nx^2}{2(2-n)t + nA^n}\right)^{-1/n} \quad$ as $x \to +\infty$;

(iii) $\alpha > \frac{2}{n}$ $\quad u_0 \sim \left(\frac{nx^2}{2(2-n)t}\right)^{-1/n} \quad$ as $x \to +\infty$ for $t > 0$.

For small t the transition to the case (iii) behaviour takes the form

$$u_0 \sim t^{\alpha/(n\alpha-2)}\sigma(xt^{1/(n\alpha-2)}) \quad \text{for } t \ll 1, \quad x = O(t^{-1/(n\alpha-2)}), \tag{4.7}$$

where $\sigma(\mu)$ satisfies

$$\left.\begin{aligned} &\frac{1}{n\alpha - 2}\left(\alpha\sigma + \mu\frac{\mathrm{d}\sigma}{\mathrm{d}\mu}\right) = \frac{\mathrm{d}}{\mathrm{d}\mu}\left(\sigma^{-n}\frac{\mathrm{d}\sigma}{\mathrm{d}\mu}\right), \\ &\text{as } \mu \to 0^+ \quad \sigma \sim A\mu^{-\alpha}, \\ &\text{as } \mu \to +\infty \quad \sigma \sim \left(\frac{n\mu^2}{2(2-n)}\right)^{-1/n}. \end{aligned}\right\} \tag{4.8}$$

The solution to (4.8) may be illustrated by the case $\alpha = 2$, $n = \frac{3}{2}$ in which it may be solved exactly to give

$$\left(\frac{A}{\sigma}\right)^{1/2} + \frac{\mu}{2}\ln\left(\frac{\mu - (A/\sigma)^{1/2}}{\mu + (A/\sigma)^{1/2}}\right) = -\frac{A^{3/2}}{2};$$

see Section 3.3 of King [14].

In addition to (4.3) there is an outer region with:

(i) $x = \varepsilon^{-1/\alpha}\hat{y}$, $u = \varepsilon w$ with $w \sim 1 + A\hat{y}^{-\alpha}$;

(ii) $x = \varepsilon^{-n/2}y$, $u = \varepsilon w$ with $w \sim w_0$ where

$$\left.\begin{aligned} &\frac{\partial w_0}{\partial t} = \frac{\partial}{\partial y}\left(w_0^{-n}\frac{\partial w_0}{\partial y}\right), \\ &\text{as } y \to 0^+ \quad w_0 \sim \left(\frac{ny^2}{2(2-n)t + nA^n}\right)^{-1/n}, \\ &\text{as } y \to +\infty \quad w_0 \to 1, \\ &\text{at } t = 0 \quad w_0 = Ay^{-2/n} + 1. \end{aligned}\right\} \tag{4.9}$$

(iii) $x = \varepsilon^{-n/2}y$, $u = \varepsilon w$ with $w \sim p(y/t^{1/2})$

where $p(\zeta)$ satisfies (3.3). As $t \to +\infty$, the solution to (4.9) satisfies $w_0 \sim p(y/t^{1/2})$.

The large-time behaviour of (4.3) may be described more completely as follows.

$$\text{(i) } u_0 \sim a^{-2/n}t^{-1/(2-n)}f(x/at^{1/(2-n)}) \quad \text{for } x = O(t^{1/(2-n)}), \tag{4.10}$$

where $f(\eta)$ is given by (1.7) and, on a longer lengthscale,

$$u_0 \sim t^{-\alpha/(2-n\alpha)}\sigma(x/t^{1/(2-n\alpha)}) \quad \text{for } x = O(t^{1/(2-n\alpha)}),$$

where $\sigma(\mu)$ satisfies

$$-\frac{1}{2-n\alpha}\left(\alpha\sigma + \mu\frac{\mathrm{d}\sigma}{\mathrm{d}\mu}\right) = \frac{\mathrm{d}}{\mathrm{d}\mu}\left(\sigma^{-n}\frac{\mathrm{d}\sigma}{\mathrm{d}\mu}\right),$$

$$\text{as } \mu \to 0^+ \quad \sigma \sim \left(\frac{n\mu^2}{2(2-n)}\right)^{-1/n},$$

$$\text{as } \mu \to +\infty \quad \sigma \sim A\mu^{-\alpha},$$

and may be illustrated by the exact solution

$$\sigma = \frac{A^2}{(A^{1/2}\mu \coth(\frac{1}{2}A^{1/2}\mu) - 2)^2}$$

which holds for $\alpha = 2$, $n = \frac{1}{2}$; see Section 3.2 of King [14].

We note that for $m > -2$ and $t = O(1)$, at leading order the mass is all contained in the high concentration region $x = O(1)$; the total mass in the low concentration regions is of $o(1)$ as $\varepsilon \to 0$.

In cases (ii) and (iii), expression (4.10) describes the large t behaviour everywhere. In all cases, on the much longer timescale $t = O(\varepsilon^{-(2-n)})$ u is again of $O(\varepsilon)$ everywhere and a fuller balance of terms occurs.

4.3. $n \geq 2$

4.3.1. $n > 2$

The asymptotic results of this section are guided by our earlier exact results. In this case (4.3) has no solution, and we must consider a shorter timescale. We write $t = \nu t^*$, where $\nu \ll 1$ is to be determined in terms of ε.

The asymptotic structure is made up of four regions as follows. Writing

$$x = s^*(t^*; \nu) + \nu z^*,$$

where s^* is determined by matching, with $s_0^*(t^*) \equiv s^*(t^*; 0)$, then we have

(I) $x = O(1)$, $x < s_0^*(t^*)$;

(II) $z^* = O(1)$;

(III) $x = O(1)$, $x > s_0^*(t^*)$;

(IV) $x = O(\varepsilon^{-1})$.

Details are as follows.

(I) We have

$$\frac{\partial u}{\partial t^*} = \nu \frac{\partial}{\partial x}\left(u^{-n} \frac{\partial u}{\partial x}\right)$$

so that

$$u \sim I(x). \tag{4.11}$$

(II) In this narrow transition layer we have $\left(\text{writing } \dot{s}^* = \dfrac{\mathrm{d}s^*}{\mathrm{d}t^*}\right)$

$$\nu \frac{\partial u}{\partial t^*} - \dot{s}^* \frac{\partial u}{\partial z^*} = \frac{\partial}{\partial z^*}\left(u^{-n} \frac{\partial u}{\partial z^*}\right),$$

so at leading order

$$\dot{s}_0^*(I(s_0^*) - u_0) = u_0^{-n} \frac{\partial u_0}{\partial z^*}, \tag{4.12}$$

where we have matched with (4.11). The arbitrary function of t^* which arises on integrating (4.12) is equivalent to a t^* dependent translation of z^* and may therefore be absorbed into the $O(\nu)$ term of $s^*(t^*; \nu)$. Further terms in the various expansions may, however, need to be determined in order to obtain s^* correctly to $O(\nu)$, in a manner somewhat similar to that of the problem discussed in Appendix 1.

Equation (4.12) implies that

$$u_0 \sim (-\dot{s}_0^*(n-1)I(s_0^*)z^*)^{-1/(n-1)} \quad \text{as } z^* \to +\infty ; \tag{4.13}$$

we note that $\dot{s}_0^* < 0$.

(III) From (4.13) it follows that in $x > s_0^*$ we should write

$$u = \nu^{1/(n-1)}\varphi$$

so that

$$\nu^{1/(n-1)} \frac{\partial \varphi}{\partial t^*} = \frac{\partial}{\partial x}\left(\varphi^{-n} \frac{\partial \varphi}{\partial x}\right).$$

Matching with (4.13) then requires that

$$\varphi_0 = (-\dot{s}_0^*(n-1)I(s_0^*)(x - s_0^*))^{-1/(n-1)} . \tag{4.14}$$

(IV) It follows from (4.14) that $u = O(\varepsilon)$ for $x = O(\nu\varepsilon^{-(n-1)})$ and the rescalings in this final region are therefore

$$u = \varepsilon w , \quad x = \nu\varepsilon^{-(n-1)}y^* ,$$

giving

$$\frac{\partial w}{\partial t^*} = \nu^{-1}\varepsilon^{n-2} \frac{\partial}{\partial y^*}\left(w^{-n} \frac{\partial w}{\partial y^*}\right),$$

which requires that we choose $\nu = \varepsilon^{n-2}$, so that $x = \varepsilon^{-1}y^*$. At leading order we have

$$\begin{aligned} &\frac{\partial w_0}{\partial t^*} = \frac{\partial}{\partial y^*}\left(w_0^{-n} \frac{\partial w_0}{\partial y^*}\right), \\ &\text{as } y^* \to 0^+ \quad w_0 \to +\infty , \\ &\text{as } y^* \to +\infty \quad w_0 \to 1 , \\ &\text{at } t^* = 0 \quad w_0 = 1 ; \end{aligned} \tag{4.15}$$

more precisely, matching with (4.15) implies that $w_0 = O(y^{*-1/(n-1)})$ as $y^* \to 0^+$. The conditions in (4.15) are sufficient to specify w_0 uniquely, and we have $w_0 = p(y^*/t^{1/2})$, where $p(\zeta)$ satisfies (3.14). Hence

$$w_0 \sim (\kappa y^*/t^{*1/2})^{-1/(n-1)} \quad \text{as } y^* \to 0^+$$

for some constant κ which is determined by solving (3.14), and from (4.14) it then follows that s_0^* is determined from

$$-\dot{s}_0^*(n-1)I(s_0^*) = \kappa t^{*-1/2},$$

which implies that

$$\int_{s_0^*}^{\infty} I(x)\,dx = \frac{2\kappa}{n-1}\, t^{*1/2}. \tag{4.16}$$

It follows that extinction of region (I) occurs at some finite t^*, $t_c^*(\varepsilon)$ say, where

$$t_c^* \sim t_{c0}^* \equiv \left(\frac{(n-1)M}{4\kappa}\right)^2, \tag{4.17}$$

and we write

$$s^*(t_c^*) = x_c, \quad s_0^*(t_{c0}^*) = x_{c0},$$

where $x_c \sim x_{c0}$ and x_{c0} is given by

$$\int_{x_{c0}}^{\infty} I(x)\,dx = \frac{M}{2}. \tag{4.18}$$

This follows because at $t^* = t_c^*$, $x = x_c$ the transition region (II) which moves in from the right meets the equivalent one moving in from the left. A schematic of the form of the solution for $t^* < t_c^*$ is illustrated in Fig. 2. The regions (II) and (III) describe the way in which mass is transferred from the high concentration region (I) to the low concentration region (IV), prior to the disappearance of region (I) at $t^* = t_c^*$. At leading order for $t^* < t_c^*$ the total mass is shared between the high concentration region (I) and the low concentration region (IV).

We note that the leading order extinction time (4.17) depends only on the total mass M and not on the details of initial distribution I(x). Because $\kappa = (n-1)\xi_0/2$, expression (4.17) is consistent with (3.13) which arose from the delta function initial conditions.

There are two other timescales of interest. Firstly we note that I$(x) = O(\varepsilon)$ for $x = O(\varepsilon^{-1/\alpha})$ and there is a shorter timescale in which we write

$$t = \varepsilon^{n-2/\alpha}\tau, \quad x = \varepsilon^{-1/\alpha}\hat{y}, \quad u = \varepsilon w$$

to obtain the leading order problem

$$\left.\begin{aligned}
&\frac{\partial w_0}{\partial \tau} = \frac{\partial}{\partial \hat{y}}\left(w_0^{-n}\frac{\partial w_0}{\partial \hat{y}}\right), \\
&\text{as } \hat{y} \to 0^+ \quad w_0 \sim A\hat{y}^{-\alpha}, \\
&\text{as } \hat{y} \to +\infty \quad w_0 \to 1, \\
&\text{at } \tau = 0 \quad\;\; w_0 = 1 + A\hat{y}^{-\alpha};
\end{aligned}\right\} \tag{4.19}$$

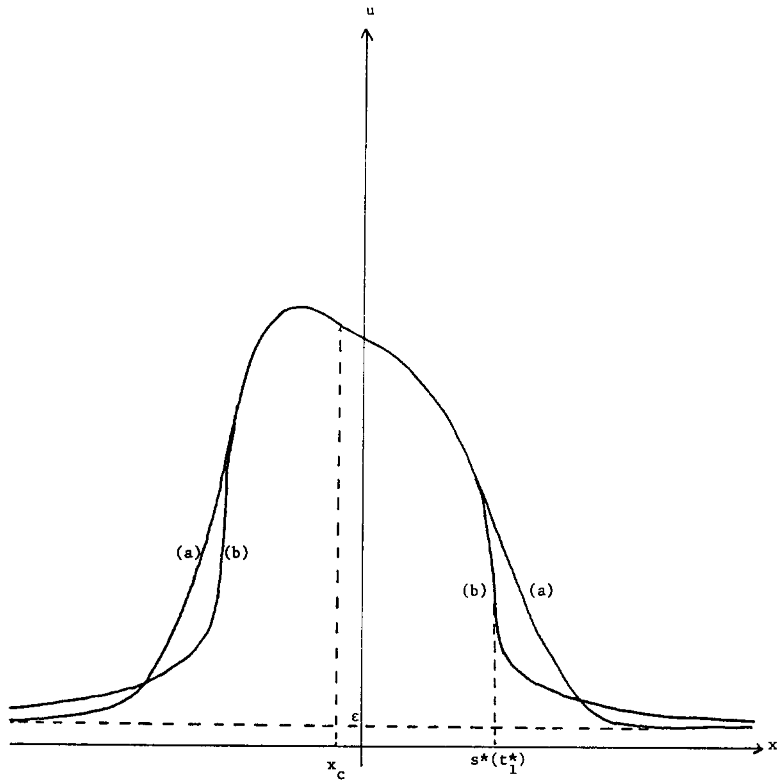


Fig. 2. Schematic of solution to (4.1) with $m<-2$, $\varepsilon \ll 1$. (a) $t^*=0$, $u=I(x)+\varepsilon$. (b) $t^*=t_1^*$ with $0<t_c^*<t_c^*$.

in $x = O(1)$

$$u \sim I(x)$$

again holds. The narrow transition region (II) first develops over the timescale $\tau = O(1)$, this development being governed by (4.19).

Secondly, there is a timescale close to the extinction time. We write

$$t^* = t_c^* + \varepsilon^{n-2}\bar{t}\,, \quad x = x_c + \varepsilon^{n-2}\bar{x}\,;$$

on this timescale regions (I) and (II) have merged into one, and at leading order we have, since $\nu = \varepsilon^{n-2}$,

$$\left.\begin{aligned} &\frac{\partial u_0}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\left(u_0^{-n}\frac{\partial u_0}{\partial \bar{x}}\right), \\ &\text{as } |\bar{x}| \to \infty \quad u_0 \sim \left(\frac{4\kappa^2}{(n-1)M}\,|\bar{x}|\right)^{-1/(n-1)}, \\ &\text{at } \bar{t} \to -\infty \quad u_0 \sim q\left(|\bar{x}| + \frac{4\kappa^2}{(n-1)^2 M I(x_{c0})}\,\bar{t}\right), \end{aligned}\right\} \tag{4.20}$$

where $q(\omega)$ satisfies

$$\frac{4\kappa^2}{(n-1)^2 M I(x_{c0})}(q - I(x_{c0})) = q^{-n}\frac{dq}{d\omega}. \tag{4.21}$$

We note that u_0 is symmetric about $\bar{x} = 0$. To completely specify the problem, the constant of integration which arises on integrating (4.20) must be specified; this corresponds to specifying the origin of $\bar{t}$ by some choice of the $O(\varepsilon^{n-2})$ term in $t_c^*(\varepsilon)$. The behaviour as $\bar{t} \to -\infty$ follows from matching, using

$$s^*(t^*) \sim x_c + (t^* - t_c^*)\dot{s}_0^*(t_{c0}^*) \quad \text{as } t^* \to t_c^{*+}$$

with

$$\dot{s}_0^*(t_{c0}^*) = -4\kappa^2/(n-1)^2 M I(x_{c0}).$$

We have in particular that

$$u_0 \sim I(x_{c0}) \text{ for } \frac{4\kappa^2(-\bar{t})}{(n-1)^2 M I(x_{c0})} - |\bar{x}| \gg 1.$$

We note the result

$$\frac{\mathrm{d}}{\mathrm{d}\bar{t}}\int_0^\infty \left(u_0 - \left(\frac{4\kappa^2}{(n-1)M}\bar{x}\right)^{-1/(n-1)}\right)\mathrm{d}\bar{x} = -\frac{4\kappa^2}{(n-1)^2 M},$$

which follows from (4.20) because the condition as $\bar{x} \to +\infty$ is equivalent to the flux condition

$$\text{as } \bar{x} \to +\infty \quad u_0^{1-n}\frac{\partial u_0}{\partial \bar{x}} \to -\frac{4\kappa^2}{(n-1)^2 M}.$$

The behaviour of (4.20) as $\bar{t} \to +\infty$ takes the form

$$u_0 \sim \bar{t}^{-1/(n-2)} b(\bar{x}/\bar{t}^{(n-1)/(n-2)}), \tag{4.22}$$

where $b(\lambda)$ satisfies

$$-\frac{1}{n-2}\left(b + (n-1)\lambda\frac{\mathrm{d}b}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(b^{-n}\frac{\mathrm{d}b}{\mathrm{d}\lambda}\right),$$

$$\text{at } \lambda = 0 \quad \frac{\mathrm{d}b}{\mathrm{d}\lambda} = 0,$$

$$\text{as } \lambda \to +\infty \quad b \sim (4\kappa^2\lambda/(n-1)M)^{-1/(n-1)}.$$

This solution is equivalent to (3.16). It follows that for $t^* = t_c^* + O(1)$ we have $u = O(\varepsilon)$, and the high concentration region has disappeared.

4.3.2. $n = 2$

It is more revealing to consider this case using asymptotic methods rather than by immediately exploiting the exact linearisability of the equation. The structure is, for the most part, similar to that for $n > 2$. The asymptotic results can, however, be motivated by, and checked against, exact solutions for special initial conditions. If, for example, we consider the case

$$I(x)\begin{cases} = \dfrac{M}{2} & |x| < 1 \\ = 0 & |x| > 1 \end{cases}$$

then writing

$$v = \int_0^x u(x', t)\,\mathrm{d}x'$$

leads to the exact solution

$$\begin{aligned} \varepsilon(M + 2\varepsilon)x = 2\varepsilon v + \frac{M}{4}\Big(&(M + 2v + 2\varepsilon)\operatorname{erfc}((M + 2v + 2\varepsilon)/4t^{1/2}) \\ &- (M - 2v + 2\varepsilon)\operatorname{erfc}((M - 2v + 2\varepsilon)/4t^{1/2}) \\ &+ 4\sqrt{\frac{t}{\pi}}\,(\exp(-(M - 2v + 2\varepsilon)^2/16t) - \exp(-(M + 2v + 2\varepsilon)^2/16t))\Big)\,, \end{aligned}$$

$$u = 2\varepsilon(M + 2\varepsilon)/(4\varepsilon + M(\operatorname{erfc}((M + 2v + 2\varepsilon)/4t^{1/2}) + \operatorname{erfc}((M - 2v + 2\varepsilon)/4t^{1/2})))\,;$$

in particular

$$u(0, t) = \varepsilon(M + 2\varepsilon)/(2\varepsilon + M\operatorname{erfc}((M + 2\varepsilon)/4t^{1/2}))\,. \tag{4.23}$$

For $n = 2$, (4.3) again has no solution so that a different balance must be found on a shorter timescale, and we write $t^* = \nu^{-1}t$ where $\nu \ll 1$ is again to be determined.

In this case five regions are required to describe the full asymptotic structure; the first four parallel those of Section 4.3.1.

(I) For $x < s_0^*(t^*)$ we have

$$u \sim I(x)\,.$$

(II) For $x = s^*(t^*; \nu) + \nu z^*$ we obtain (4.12) with $n = 2$, so that

$$-\dot{s}_0^* z^* = \frac{1}{I(s_0^*)u_0^*} + \frac{1}{I^2(s_0^*)}\ln((I(s_0^*) - u_0)/u_0)\,.$$

It follows that

$$u_0 \sim -1/\dot{s}_0^* I(s_0^*)z^* \quad \text{as } z^* \to +\infty\,, \tag{4.24}$$

which corresponds to (4.13).

(III) For $x = s_0^* + O(1)$ we write $u = \nu\varphi$ to give

$$\nu \frac{\partial \varphi}{\partial t^*} = \frac{\partial}{\partial x}\left(\varphi^{-2}\frac{\partial \varphi}{\partial x}\right),$$

and matching with (4.24) implies that

$$\varphi_0 = -1/\dot{s}_0^* I(s_0^*)(x - s_0^*) . \tag{4.25}$$

In this case an intermediate region (region (V)) is required between regions (III) and (IV), but in order to motivate this we first discuss region (IV).

(IV) Writing

$$u = \varepsilon w , \quad x = \nu^{1/2}\varepsilon^{-1} y^*$$

yields at leading order (4.15) with $n = 2$, so that w_0 is given parametrically by (see (3.9))

$$\left.\begin{aligned} \zeta &= \frac{1}{\sqrt{\pi}}\, e^{-\xi^2/4} - \frac{\xi}{2}\operatorname{erfc}(\xi/2) , \\ w_0 &= 2/\operatorname{erfc}(\xi/2) , \end{aligned}\right\} \tag{4.26}$$

where

$$\zeta = y^*/t^{*1/2} .$$

From (4.26) it follows that

$$w_0 \sim \frac{1}{\zeta \ln^{1/2}(1/\zeta)}\left(1 + \frac{\ln\ln(1/\zeta)}{2\ln(1/\zeta)} + \frac{\ln(2\sqrt{\pi}) - 2}{2\ln(1/\zeta)}\right) \text{as } \zeta \to 0^+ , \tag{4.27}$$

and the presence of the $\ln(1/\zeta)$ terms makes the matching rather more delicate than for $n > 2$, and prevents direct matching of (4.27) with (4.25). In the intermediate region we write

(V) $w = \nu^{1/2}\rho/y^*$, $\quad x^* = \nu \ln(1/y^*)$

to give

$$\frac{\partial \rho}{\partial t^*} + \rho^{-2}\frac{\partial \rho}{\partial x^*} = \nu \frac{\partial}{\partial x^*}\left(\rho^{-2}\frac{\partial \rho}{\partial x^*}\right).$$

Matching with (4.27) requires that

$$\rho_0 \sim (t^*/x^*)^{1/2} \quad \text{as } x^* \to 0^+ , \tag{4.28}$$

and, since

$$\frac{\partial \rho_0}{\partial t^*} + \rho_0^{-2}\frac{\partial \rho_0}{\partial x^*} = 0 ,$$

this implies that

$$\rho_0 = (t^* / x^*)^{1/2} . \tag{4.29}$$

Because we have

$$\rho = x\varphi , \quad x = \nu^{1/2}\varepsilon^{-1} e^{-x^*/\nu} ,$$

it follows that (4.29) may be written as

$$\rho_0 = t^{*1/2}/(\nu \ln(\nu^{1/2}/\varepsilon) - \nu \ln x)^{1/2} ,$$

and matching with (4.25) requires that

$$\nu \ln(\nu^{1/2}/\varepsilon) = O(1) .$$

We may then take

$$\nu = 1/\ln(1/\varepsilon) ,$$

which gives

$$x^* = 1 - \tfrac{1}{2}\nu \ln(1/\nu) - \nu \ln x .$$

Expression (4.25) then also requires that

$$-\dot{s}_0^* I(s_0^*) = t^{*-1/2}$$

so that

$$\int_{s_0^*}^{\infty} I(x)\, dx = 2t^{*1/2} ;$$

this may be compared to (4.16). Extinction occurs at $t^* = t_c^*(\varepsilon)$, $x = x_c(\varepsilon)$ where

$$t_c^* \sim t_{c0}^* \equiv M^2/16 , \quad x_c \sim x_{c0} ,$$

with x_{c0} given by (4.18).

There is again a shorter timescale described by (4.19) which we do not discuss further. Close to the extinction time we introduce the rescalings

$$t^* = t_c^* + \nu\bar{t} , \quad x = x_c + \nu\bar{x}$$

to give (4.20) and (4.21) with $n = 2$, $\kappa = 1$. In this case

$$\dot{s}_0^*(t_{c0}) = -4/MI(x_{c0})$$

holds.

When $n=2$ we may solve (4.20) exactly by means of the non-local transformation, and this case illustrates the general case. We write

$$v_0 = \int_0^{\bar{x}} u_0(x', \bar{t})\,\mathrm{d}x'$$

to give, for $\bar{x} \geq 0$,

$$\left.\begin{aligned} &\frac{\partial v_0}{\partial \bar{t}} = \left(\frac{\partial v_0}{\partial \bar{x}}\right)^{-2} \frac{\partial^2 v_0}{\partial \bar{x}^2}\,, \\ &\text{at } \bar{x}=0 \quad v_0 = 0\,, \\ &\text{as } \bar{x} \to +\infty \quad v_0 \sim \frac{M}{4} \ln \bar{x}\,, \\ &\text{as } \bar{t} \to -\infty \quad \frac{v_0}{I(x_{c0})} + \exp\left(\frac{4}{M}\left(v_0 + \frac{4\bar{t}}{M}\right)\right) \sim \bar{x}\,. \end{aligned}\right\} \tag{4.30}$$

To deduce this behaviour as $\bar{t} \to -\infty$ we note that it follows from (4.20)–(4.21) with $n=2$, $\kappa = 1$ that

$$v_0 \sim -\frac{4\bar{t}}{M} + r\left(\bar{x} + \frac{4}{MI(x_{c0})}\,\bar{t}\right) \quad \text{as } \bar{t} \to -\infty\,,$$

where

$$q(\omega) = \frac{\mathrm{d}r}{\mathrm{d}\omega}(\omega)\,,$$

with

$$\frac{4}{MI(x_{c0})}\left(1 - I(x_{c0})\,\frac{\mathrm{d}\omega}{\mathrm{d}r}\right) = -\frac{\mathrm{d}^2\omega}{\mathrm{d}r^2}\,.$$

Hence, because $v_0 = 0$ at $\bar{x} = 0$, we obtain

$$\omega = \frac{r}{I(x_{c0})} + Ce^{(4/M)r}$$

for some constant C, and we may set $C=1$ by choice of the origin of $\bar{t}$ to give the condition appearing in (4.30).

The solution to (4.30) may readily be obtained in the form

$$\bar{x} = \frac{v_0}{I(x_{c0})} + 2\,\mathrm{e}^{16\bar{t}/M^2} \sinh(4v_0/M)\,,$$

so that

$$u_0 = MI(x_{c0})/(M + 8I(x_{c0})\,\mathrm{e}^{16\bar{t}/M^2} \cosh(4v_0/M))\,. \tag{4.31}$$

It follows that

$$u_0 \sim \frac{M}{4(\bar{x}^2 + 4\,\mathrm{e}^{32\bar{t}/M^2})^{1/2}} \quad \text{as } \bar{t} \to +\infty\,, \tag{4.32}$$

which may be written in the self-similar form

$$u_0 \sim \mathrm{e}^{-16\bar{t}/M^2} h(\bar{x}/\mathrm{e}^{16\bar{t}/M^2})\,, \tag{4.33}$$

with

$$h(\lambda) = M/4(\lambda^2 + 4)^{1/2}\,.$$

This is an exact solution to (1.1) with $m = -2$, and is the solution which corresponds to (1.5)–(1.7) with $m > -2$; see King [14]. We also note the result

$$u_0(0, \bar{t}) = MI(x_{c0})/(M + 8I(x_{c0})\,\mathrm{e}^{16\bar{t}/M^2})\,, \tag{4.34}$$

which follows from (4.31).

Because, in contrast to (4.22), the decay of (4.33) is exponential rather than algebraic in $\bar{t}$, we are unable to match (4.32) directly into the late-stage development which occurs when $u = O(\varepsilon)$ everywhere. The situation can be clarified by examining the special case (4.23) in which

$$u(0, t) \sim M/(2 + M\,\mathrm{erfc}(M/4t^{1/2})/\varepsilon)$$

holds as $\varepsilon \to 0$, so that for $t \ll 1$

$$u(0, t) \sim M/(2 + 4t^{1/2}\exp(-M^2/16t)/\sqrt{\pi}\varepsilon)\,. \tag{4.35}$$

Hence, writing $\nu = 1/\ln(1/\varepsilon)$ as before, for

$$t^* = t_c^* + \nu\bar{t}\,,$$

with

$$t_c^* = \frac{M^2}{16}\left(1 + \frac{\nu}{2}\ln(1/\nu) - \nu\ln(M/8\sqrt{\pi})\right),$$

we obtain

$$u(0, t) \sim M/(2 + 4\,\mathrm{e}^{16\bar{t}/M^2})$$

(the $O(\nu)$ term in t_c^* is chosen for consistency with (4.34)). It is evident from (4.35) that the late-stage behaviour in which $u(0, t) = O(\varepsilon)$ occurs for $t = O(1)$, and not for $t^* = t_c^* + O(1)$, which is the case for $n > 2$.

If we consider the timescale $t = O(1)$ with

$$u = \varepsilon w\,, \quad x = \varepsilon^{-1}y\,,$$

then we have

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial y}\left(w^{-2}\frac{\partial w}{\partial y}\right), \tag{4.36}$$

and the leading order initial condition on (4.36) (which comes from matching back into the shorter time behaviour) may be written

$$\text{at } t = 0 \quad w = M\delta(y) + 1 .$$

We may therefore use the solution of Section 4.3.2. This implies that for $t \ll 1$, $y = O(t^{1/2})$, we have

$$w \sim w_0(y/t^{1/2}) ,$$

where $w_0(\zeta)$ is again given by (4.26). The solution in the high concentration regime follows from (3.10), so that for $t \ll 1$, $y = O(t^{3/2}\exp(-M^2/16t))$, we have

$$w \sim 4t/(M^2y^2 + 256t^3\,e^{-M^2/8t}/\pi M^2)^{1/2} . \tag{4.37}$$

Since $t = \nu t_c^* + \nu^2\bar{t}$ we have for $\bar{t} = O(1)$

$$\frac{1}{t} \sim \frac{1}{\nu t_c^*} - \frac{\bar{t}}{t_c^{*2}}$$

and because $t_c^* \sim M^2/16$ we recover from (4.37) the $\exp(32\bar{t}/M^2)$ dependence occurring in (4.32), with which it is possible to match (4.37).

5. Discussion

As already noted, when $m \leq -2$ the equation (1.1) has no finite mass solutions valid on an infinite domain. One aim of this paper was to gain understanding of the limiting processes in which the width of the domain tends to infinity or in which the background concentration tends to zero. Such limits are of relevance to, in particular, semiconductor applications, as is the surface source problem (discussed in Appendix 2) which has a number of similar features.

This paper was also intended to extend the results of [12], [13] and [11] for diffusivities of the representative form

$$D(c) = (c + \varepsilon)^m$$

(with $\varepsilon \ll 1$ and with $c \to 0$ as $|x| \to \infty$) by considering the full range of values of m. The following particular features of such problems may be highlighted.

(a) For $m > 0$ the solution for $\varepsilon = 0$ has compact support if the initial data does; for $\varepsilon > 0$ it does not.
(b) For $m < -2$ there is no solution when $\varepsilon = 0$.

There are a number of ways in which the analysis of this paper could be extended. If, for example, we consider general diffusivities $D(u)$, so that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\,\frac{\partial u}{\partial x}\right),$$

and impose conditions (1.9) with $\varepsilon = 1$, then under the transformation (3.6) we obtain

$$\frac{\partial U}{\partial T} = \frac{\partial}{\partial X}\left(D^*(U)\,\frac{\partial U}{\partial X}\right), \tag{5.1}$$

with $D^*(U) = D(1/U)/U^2$, subject to the same boundary and initial conditions as in (3.5). If, for sufficiently small U, we have

$$\int_0^U (D^*(U')/U')\,\mathrm{d}U' < \infty, \tag{5.2}$$

(this is the criterion for the finite speed of propagation of the interface between regions in which $U > 0$ and in which $U \equiv 0$) then the solution to (5.1) may be written exactly for finite time in the form (3.11), where $P(\xi)$ satisfies the free boundary problem

$$-\tfrac{1}{2}\xi\,\frac{\mathrm{d}P}{\mathrm{d}\xi} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left(D^*(P)\,\frac{\mathrm{d}P}{\mathrm{d}\xi}\right),$$

$$\text{as } \xi \to -\infty \quad P \to 1,$$

$$\text{at } \xi = \xi_0 \qquad P = 0, \quad \lim_{\xi\to\xi_0}\left(\frac{D^*(P)}{P}\,\frac{\mathrm{d}P}{\mathrm{d}\xi}\right) = -\tfrac{1}{2}\xi_0,$$

where ξ_0 must be determined as part of the solution. The form (3.11) is again valid for $T < T_c$ where T_c is given by (3.13).

The condition (5.2) is equivalent to

$$\int_u^\infty u' D(u')\,\mathrm{d}u' < \infty \tag{5.3}$$

for sufficiently large u, and this is therefore the condition for the persistence of the delta function for finite time. The behaviour at $x = 0$ is again given by (3.15). The condition (5.3) can obviously be satisfied by diffusivities, such as $D(u) = \mathrm{e}^{-u}$, which are not singular as $u \to 0$.

If initial conditions (incorporating a delta function) are chosen such that the corresponding solution to (5.1) is a "waiting-time" solution (see Lacey et al. [15]) in which the support of U does not change until $T = T_W$ say, with $T_W > 0$, then the amplitude of the delta function will be undiminished for $t \leq T_W$ after which it will decrease.

It can also be instructive to consider more general forms of initial-boundary value problems for (1.1). One such generalisation would be to impose initial conditions with

$$\text{as } x \to -\infty \quad u \to \varepsilon_1, \quad \text{as } x \to +\infty \quad u \to \varepsilon_2,$$

for different constant background concentrations ε_1 and ε_2. A particular consequence of this change would be that, in the problem corresponding to (4.20), u_0 would not be symmetric.

The values of the background concentrations play a decisive role in determining the rate of loss of mass from the high concentration region.

Another generalisation, which combines some of the features discussed in Sections 2 and 3, is the following:

$$\begin{aligned}
&\text{at } x=0 && \frac{\partial u}{\partial x}=0\,,\\
&\text{as } x\to+\infty && u\to 1\,,\\
&\text{at } t=0 && u=M_1\delta(x)+\tfrac{1}{2}M_2\delta(x-x_0)+H(x-x_0)\,,
\end{aligned}$$

where $x_0>0$; we take

$$\int_0^\infty \delta(x)\,\mathrm{d}x=\frac{1}{2}\,.$$

Transformed conditions are then

$$\begin{aligned}
&\text{at } X=0 && \frac{\partial U}{\partial X}=0\,,\\
&\text{as } X\to+\infty && U\to 1\,,\\
&\text{at } T=0 && U=x_0\delta\left(X-\frac{M_1}{2}\right)+H(X-\tfrac{1}{2}(M_1+M_2))\,.
\end{aligned}$$

For $n>2$ the solution for U is thus given exactly for some finite time $T<T_c$, say, by an instantaneous source solution centered at $X=M_1/2$ together with a solution of the form

$$U=P((M_1+M_2-2X)/2T^{1/2})$$

which will hold for sufficiently large X; $P(\xi)$ again satisfies (3.12). It may then be shown that, for $t<T_c$, u takes the form

$$u=t^{1/n}g(x)\quad \text{for } 0<x<x_0\,,$$

with $g\to+\infty$ as $x\to 0^+$ and as $x\to x_0^-$, and

$$u=p((x-x_0)/t^{1/2})\quad \text{for } x>x_0\,,$$

with $p(\zeta)\to+\infty$ as $\zeta\to 0^+$, with delta functions persisting at $x=0$ and at $x=x_0$. Since $U=0$ for $0\leq X\leq \tfrac{1}{2}M_1-aT^{1/n}$ and for $\tfrac{1}{2}M_1+aT^{1/n}\leq X\leq \tfrac{1}{2}(M_1+M_2)-\xi_0T^{1/2}$, for some constants a and ξ_0, it in fact follows that

$$\text{at } x=0\quad u=(M_1-2at^{1/n})\delta(x)$$

and

$$\text{at } x=x_0\quad u=\tfrac{1}{2}(M_2-2at^{1/n}-2\xi_0t^{1/2})\delta(x-x_0)\,,$$

as long as the coefficients of both delta functions are positive.

This last example illustrates some of the types of similarity solution which play a role in these problems, which include the instantaneous source solution (1.5), the separable solution $u = t^{1/n}g(x)$ and the Boltzmann solution $u = p(x/t^{1/2})$. We may summarise the occurrence of such solutions in Sections 2 and 3 as follows.

(1) $m > 0$ Instantaneous source solution.

(2) $-2 < m < 0$. Instantaneous source solution at high concentrations; either separable solution (Section 2) or Boltzmann solution (Section 3) at lower concentrations.

(3) $m < -2$ Separable solution (Section 2) or Boltzmann solution (Section 3).

We emphasise that in cases (2) and (3) the low concentration behaviour depends strongly on the form of the boundary conditions. The leading order high concentration behaviour in cases (1) and (2) is, for sufficiently small times, independent of which boundary conditions hold, whereas for case (3) it is not, the rate of depletion of the delta function (or, as in Section 4, the high concentration region) being determined by the low concentration behaviour. Roughly speaking, the behaviour is dictated by the high concentration region in case (1) and by the low concentration regions in case (3), while in case (2) the details of the high and low concentration regions have little influence on one another.

There is a further similarity solution which plays an important role for $m < -2$, namely the solution (4.22). The instantaneous source solution for $m > -2$ is often thought of as the solution which describes the behaviour once the details of the initial conditions have been forgotten. The solution (4.22) plays a similar role here in the case $m < -2$. As may be seen by comparison of Sections 3 and 4, once (4.22) comes to describe the behaviour the solution for delta function initial conditions approximates that for much more general initial data; the same holds true regarding a generalisation of the problem discussed in Section 2 whereby the mass is initially concentrated in a region of thickness much less than $L_1 + L_2$. One respect in which the solution (4.22) differs from the instantaneous source solution is that it describes the behaviour fairly close to the particular time $t^* = t_c^*$, whereas the latter gives the behaviour of (4.3) as $t \to \infty$. Nevertheless, the solution (4.22) is that which, for $m < -2$, plays the role closest to that of an instantaneous source solution in describing the behaviour at high concentrations.

We note that in the borderline case $m = -2$ the relevant solution (either (2.13) or (3.10)) has a rather unusual time dependence which depends (slightly) on which boundary conditions hold; these expressions take the self-similar forms

$$u \sim t^{1/2}\,e^{M^2/16t} f(x/t^{1/2}\,e^{-M^2/16t})$$

and

$$u \sim t^{-1/2}\,e^{M^2/16t} f(x/t^{3/2}\,e^{-M^2/16t})$$

respectively. For $m > -2$ the solution is of the same form in each case (see (2.18) and (3.16)).

As already indicated, for all values of m the solution for delta function initial conditions is approached (on appropriate timescales) by the solutions for much more general initial data. This provides justification for considering problems with delta function initial conditions. The appropriate timescales in Section 4 are those for which $\varepsilon \ll u_{\max} \ll 1$. When u has become of $O(\varepsilon)$ everywhere the instantaneous source solution for $m > -2$ and the solution (4.22) for $m < -2$ no longer describe the behaviour. These solutions therefore provide a description of the intermediate asymptotics of the problem in the sense discussed by Barenblatt [16].

We have limited this paper to discussing one-dimensional problems. Additional effects can arise in higher dimensions and such matters will be addressed elsewhere.

Appendix 1. Asymptotic analysis of (3.1) with $m > 0$

In this appendix we briefly discuss the behaviour of (3.1) for $0 < \varepsilon \ll 1$, $m > 0$. The case $m = 1$ has been discussed in King and Please [12] and King [13]; this turns out to be a borderline case, and here we shall discuss the cases $m < 1$ and $m > 1$.

The asymptotic structure for (3.1) consists of two outer regions $x < s$ and $x > s$, where $s(t;\ \varepsilon)$ is to be determined, separated by a narrow transition region in which $x = s + O(\varepsilon^m)$.

(1) Transition region

Introducing inner variables

$$x = s(t;\ \varepsilon) + \varepsilon^m z,\ u = \varepsilon w$$

we have

$$\varepsilon^m \frac{\partial w}{\partial t} - \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left(w^m \frac{\partial w}{\partial z}\right).$$

Writing $s = s_0(t) + o(1)$, $w = w_0(z, t) + o(1)$ as $\varepsilon \to 0$ we obtain $\left(\text{writing } \dot{s}_0 \equiv \frac{\mathrm{d}s_0}{\mathrm{d}t}\right)$

$$-\dot{s}_0(w_0 - 1) = w_0^m \frac{\partial w_0}{\partial z}, \qquad \text{(A1.1)}$$

where we have imposed $w_0 \to 1$ as $z \to +\infty$ because $u - \varepsilon$ is exponentially small in the second outer region $x > s$. The arbitrary function of t which arises on integrating (A1.1) corresponds to a translation in the origin of z and may without loss of generality be absorbed into the $O(\varepsilon^m)$ term of $s(t;\ \varepsilon)$. Hence we may write

(a) m integer

$$-\dot{s}_0 z = \sum_{k=1}^{m} \frac{w_0^k}{k} + \ln(w_0 - 1) \qquad \text{(A1.2)}$$

(b) m non-integer

$$-\dot{s}_0 z = \sum_{k=0}^{\infty} \frac{w_0^{m-k}}{m-k} \quad \text{for } w_0 > 1, \qquad \text{(A1.3)}$$

with

$$-\dot{s}_0 z = \ln(w_0 - 1) + \frac{1}{m} + \sum_{k=1}^{\infty} \frac{m}{k(m-k)} + o(1) \quad \text{as } w_0 \to 1^+ ; \tag{A1.4}$$

we note that

$$\sum_{k=1}^{\infty} \frac{m}{k(m-k)} = \psi(1-m) + \gamma ,$$

where

$$\psi(z) = \frac{d}{dz} (\ln \Gamma(z))$$

is the digamma function and γ is Euler's constant.

From (A1.2) and (A1.3) it follows that if $m \neq 1$ then

$$w_0 \sim (-m\dot{s}_0 z)^{1/m} - \frac{1}{m-1} \quad \text{as } z \to -\infty . \tag{A1.5}$$

(2) First outer region $(x < s)$

In $x < s$ we write

$$u \sim u_0(x, t) + \varepsilon u_1(x, t)$$

and it follows from (3.1) that u_0 is given by (1.5) and (1.6), and matching with (A1.5) then requires that

$$s_0 = at^{1/(m+2)} . \tag{A1.6}$$

The correction term u_1 satisfies

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2}{\partial x^2} (u_0^m u_1)$$

so that

$$(m+2)t \frac{\partial u_1}{\partial t} - \eta \frac{\partial u_1}{\partial \eta} = \frac{m}{2} \frac{\partial^2}{\partial \eta^2} ((1-\eta^2)u_1) , \tag{A1.7}$$

where

$$\eta = x/at^{1/(m+2)} .$$

The behaviour now depends on whether $m - 1$ is positive or negative. If $m < 1$ we can match directly into (A1.5) by writing $u_1 \equiv u_1(\eta)$ so that u_1 satisfies

$$m(1-\eta^2) \frac{d^2 u_1}{d\eta^2} - 2(2m-1)\eta \frac{du_1}{d\eta} - 2mu_1 = 0 , \tag{A1.8}$$

$$\left.\begin{aligned} &\text{at } \eta = 0 \quad \frac{\mathrm{d}u_1}{\mathrm{d}\eta} = 0\,, \\ &\text{at } \eta = 1 \quad u_1 = -\frac{1}{m-1}\,. \end{aligned}\right\} \tag{A1.9}$$

The solution may be written

$$u_1 = \sum_{k=0}^{\infty} b_k \eta^{2k} \tag{A1.10}$$

where

$$b_{k+2} = \frac{2mk^2 + (3m-2)k + m}{m(k+1)(2k+1)}\, b_k\,, \tag{A1.11}$$

and b_0 is chosen so that

$$\sum_{k=1}^{\infty} b_k = -\frac{1}{m-1}\,. \tag{A1.12}$$

We note that $b_k = O(k^{-1/m})$ as $k \to \infty$ so that the sum in (A1.12) is only convergent for $m < 1$. Equivalently, it may be shown that the solution (A1.10) and (A1.11) behaves as follows:

$$\begin{aligned} &\text{if } m < 1 \quad u_1 = O(1) \qquad \text{as } \eta \to 1^-\,; \\ &\text{if } m > 1 \quad u_1 = O((1-\eta)^{(1-m)/m}) \quad \text{as } \eta \to 1^-\,. \end{aligned}$$

For $m < 1$, matching with (A1.5) requires that

$$s = s_0(t) + o(\varepsilon^m)\,. \tag{A1.13}$$

For $m > 1$, (A1.8) subject to (A1.9) has no solution, and in place of (A1.13) we have

$$s \sim s_0(t) + \varepsilon s_1(t)$$

so that (A1.5) provides the matching condition

$$u_1 \sim (m(s_0 - x))^{(1-m)/m} \dot{s}_0^{1/m} s_1 - \frac{1}{m-1} \quad \text{as } x \to s_0^-\,.$$

We must therefore solve (A1.8) subject to

$$\begin{aligned} &\textit{at } \eta = 0 \quad \frac{\mathrm{d}u_1}{\mathrm{d}\eta} = 0\,, \\ &\text{as } \eta \to 1^- \quad u_1 = \alpha(1-\eta)^{(1-m)/m} - \frac{1}{m-1} + o(1)\,, \end{aligned}$$

where the constant α is determined as part of solution. The solution again takes the form given by (A1.10) and (A1.11), and $s_1(t)$ is given by

$$s_1 = \alpha(ms_0)^{(m-1)/m}\dot{s}_0^{-1/m}$$

so that

$$s_1 = \alpha a^{(m-2)/m} m\left(\frac{m+2}{m}\right)^{1/m} t^{2/(m+2)} .$$

However, to obtain s correctly to $O(\varepsilon^m)$ further correction terms may be needed, depending on the value of m. For $1 < m < 2$ the preceding analysis is sufficient, and

$$s = s_0(t) + \varepsilon s_1(t) + o(\varepsilon^m) ;$$

for $m = 2$ terms in s of $O(\varepsilon^2 \ln(1/\varepsilon))$ and $O(\varepsilon^2)$ are needed, and the term in u of $O(\varepsilon^2)$ must be calculated. In general increasing m by one increases the number of terms needed.

The significance of these correction terms lies in the fact that a leading order expression for $u - \varepsilon$ in the second outer region cannot be determined unless s is determined up to $O(\varepsilon^m)$. This follows from (A1.2) and (A1.4) which imply that

$$w_0 \sim 1 + Ae^{-\dot{s}_0 z} \quad \text{as } z \to +\infty ,$$

where the constant A can be determined from (A1.2) or (A1.4). Written in terms of the outer variables this reads

$$u - \varepsilon \sim A\varepsilon e^{-\dot{s}_0(x-s)/\varepsilon^m} , \tag{A1.14}$$

so that terms in s up to $O(\varepsilon^m)$ must be known for the leading order term in $u - \varepsilon$ in $x > s$ to be calculated.

(3) *Second outer region* $(x > s)$

Since $u - \varepsilon$ is exponentially small in $x > s$, we can write

$$\frac{\partial u}{\partial t} \sim \varepsilon^m \frac{\partial^2 u}{\partial x^2} ,$$

and writing

$$\ln(u - \varepsilon) \sim -F_0(x, t)/\varepsilon^m$$

(we shall only consider the leading order term in $\ln(u - \varepsilon)$) we have

$$\begin{aligned} &\frac{\partial F_0}{\partial t} = -\left(\frac{\partial F_0}{\partial x}\right)^2 , \\ &\text{at } x = s_0 \qquad F_0 = 0 , \quad \frac{\partial F_0}{\partial x} = \dot{s}_0 , \end{aligned} \tag{A1.15}$$

where we have matched with (A1.14). Because s_0 is given by (A1.6), F_0 takes the self-similar form

$$F_0 = a^2 t^{-m/(m+2)} G_0(\eta) ,$$

where $\eta = x/at^{1/(m+2)}$ and

$$\left.\begin{aligned} &\frac{1}{m+2}\left(mG_0 + \eta\,\frac{\mathrm{d}G_0}{\mathrm{d}\eta}\right) = \left(\frac{\mathrm{d}G_0}{\mathrm{d}\eta}\right)^2 , \\ &\text{at } \eta = 1 \quad G_0 = 0 , \quad \frac{\mathrm{d}G_0}{\mathrm{d}\eta} = \frac{1}{m+2} . \end{aligned}\right. \tag{A1.16}$$

By writing $G_0(\eta) = \eta^2 H_0(\eta)$ we may transform (A1.16) into a separable equation and we may write its solution in the form

$$(\eta + \sqrt{\eta^2 + 4m(m+2)G_0})((m+1)\eta - \sqrt{\eta^2 + 4m(m+2)G_0})^{m+1} = 2m^{m+1} .$$

We note that for any m

$$F_0 \sim x^2/4t \quad \text{as } x \to +\infty ,$$

corresponding to the far-field behaviour for linear diffusion.

Appendix 2. The surface source problem

In this appendix we consider another initial-boundary value problem which is of practical relevance, namely

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-n}\,\frac{\partial u}{\partial x}\right) , \\ &\text{at } x = 0 \qquad u = 1 , \\ &\text{as } x \to +\infty \quad u \to \varepsilon , \\ &\text{at } t = 0 \qquad u = \varepsilon . \end{aligned}\right\} \tag{A2.1}$$

The solution to (A2.1) takes the self-similar form $u = u(x/t^{1/2})$. For $n < 2$ the problem (A2.1) has a solution when $\varepsilon = 0$; for $n \geq 2$ it does not, and here we briefly discuss this case in the limit $\varepsilon \to 0^+$. Seeger [17] has given the exact solution for $n = 2$, but an asymptotic approach is nevertheless of value even then because $\varepsilon \ll 1$ is a physically important case and the exact solution is fairly complicated.

The asymptotic behaviour for $\varepsilon \ll 1$ and $n \geq 2$ is as follows.

(a) $n > 2$

The asymptotic solution has two regions, namely $x = O(\varepsilon^{-n/2})$ and $x = O(\varepsilon^{-(n-2)/2})$. Writing

$$u = \varepsilon w , \quad x = \varepsilon^{-n/2} y ,$$

we have outer solution

$$w \sim p(y/t^{1/2}) ,$$

where $p(\zeta)$ satisfies (3.14), giving

$$p(\zeta) \sim (\kappa\zeta)^{-1/(n-1)} \quad \text{as } \zeta \to 0^+ ,$$

where the constant κ is determined as part of the solution to (3.14).

In the inner (surface) region where $u = O(1)$ we write

$$x = \varepsilon^{(n-2)/2}\hat{x}$$

and the leading order inner solution is quasi-steady:

$$u \sim (1 + \kappa\hat{x}/t^{1/2})^{-1/(n-1)} . \tag{A2.2}$$

(*b*) $n = 2$

Writing

$$u = \varepsilon w , \quad x = \varepsilon^{-1} y ,$$

the leading order outer solution is given by (4.26) with $\zeta = y/t^{1/2}$. In terms of these variables (5.1) reads

$$\left.\begin{aligned} &-\tfrac{1}{2}\zeta \frac{dw}{d\zeta} = \frac{d}{d\zeta}\left(w^{-2}\frac{dw}{d\zeta}\right) , \\ &\text{at } \zeta = 0 \qquad w = 1/\varepsilon , \\ &\text{as } \zeta \to +\infty \quad w \to 1 . \end{aligned}\right\} \tag{A2.3}$$

The surface region in which $u = O(1)$ is given by

$$\zeta = \nu^{1/2}\varepsilon\hat{\zeta}$$

where $\nu = 1/\ln(1/\varepsilon)$ so that

$$\frac{d}{d\hat{\zeta}}\left(u^{-2}\frac{du}{d\hat{\zeta}}\right) = -\tfrac{1}{2}\nu\hat{\zeta}\frac{du}{d\hat{\zeta}} . \tag{A2.4}$$

In order to match the solutions for $\hat{\zeta} = O(1)$ and $\zeta = O(1)$ we require a transition region in which we write

$$w = \nu^{1/2}\rho^*/\zeta , \quad \zeta^* = \nu \ln(1/\zeta) ,$$

giving

$$\rho^{*-2}\frac{d\rho^*}{d\zeta^*}+\tfrac{1}{2}\rho^*=\nu\frac{d}{d\zeta^*}\left(\rho^{*-2}\frac{d\rho^*}{d\zeta^*}\right)-\tfrac{1}{2}\nu\frac{d\rho^*}{d\zeta^*},$$

and matching with (4.26) using (4.27) we may obtain

$$\rho^*\sim\frac{1}{\zeta^{*1/2}}\left(1+\nu\ln(1/\nu)\frac{1}{2\zeta^*}+\nu\frac{1}{2\zeta^*}(\ln(2\sqrt{\pi}\zeta^*)-2)\right). \tag{A2.5}$$

From (5.4) we may then derive for $\hat{\zeta}=O(1)$

$$u\sim\frac{1}{(1+\hat{\zeta})}\left(1+\nu\ln(1/\nu)\frac{\hat{\zeta}}{4(1+\hat{\zeta})}+\nu\tfrac{1}{2}\left(\ln(1+\hat{\zeta})+\frac{(\ln(2\sqrt{\pi})-2)\hat{\zeta}}{(1+\hat{\zeta})}+\frac{\ln(1+\hat{\zeta})}{(1+\hat{\zeta})}\right)\right),$$

where we have matched with (A2.5), using

$$u=\rho^*/\hat{\zeta}\,,\quad \zeta^*=1+\tfrac{1}{2}\nu\ln(1/\nu)-\nu\ln\hat{\zeta}\,.$$

In terms of the original variables the leading order solution in the surface region thus reads

$$u\sim 1/(1+\ln^{1/2}(1/\varepsilon)x/t^{1/2})$$

and this is of the form noted by Gösele et al. [5] (their equation (9)) as providing a good fit to an experimental profile. More generally we may note that the solution (A2.2) describing the important high concentration region is particularly simple and is well-suited to fitting to experimental data.

We note that the exact solution for $n=2$ may be written in the form

$$\zeta=\varepsilon\gamma e^{(\gamma^2-\xi^2)/4}-\xi\left(1-(1-\varepsilon)\frac{\operatorname{erfc}(-\xi/2)}{\operatorname{erfc}(-\gamma/2)}\right),$$

$$w=1\Big/\left(1-(1-\varepsilon)\frac{\operatorname{erfc}(-\xi/2)}{\operatorname{erfc}(-\gamma/2)}\right),$$

where the constant γ is determined from

$$\frac{\gamma}{2}e^{\gamma^2/4}\operatorname{erfc}(-\gamma/2)=\frac{1}{\sqrt{\pi}}\left(\frac{1}{\varepsilon}-1\right).$$

The solution is thus expressed in terms of a parameter ξ, with

$$\xi=\int_\zeta^\infty(w(\zeta')-1)\,d\zeta'-\zeta\,;$$

we have

$$\int_0^\infty(w(\zeta)-1)\,d\zeta=\gamma$$

and

$$\gamma\sim\frac{2}{\nu^{1/2}}(1-\tfrac{1}{4}\nu\ln(1/\nu)-\tfrac{1}{2}\nu\ln(2\sqrt{\pi}))\quad\text{as }\varepsilon\to 0\,.$$

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